4 Basis and dimension

Recall that $\mathcal{S}$ is a spanning set for a space $\mathcal{V}$ if and only if every vector in $\mathcal{V}$ is a linear combination of vectors[4] in S. However, spanning sets can contain redundant vectors.

Basis A linearly independent spanning set for a vector space $\mathcal{V}$ is called a **basis** for $\mathcal{V}$.

1. Let $\mathcal{V}$ be a subspace of $\mathbb{R}^m$, and let $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, ..., \boldsymbol{b}_n\} \subseteq \mathcal{V}$. (a) Show that if $\mathcal{B}$ is a minimal spanning set for $\mathcal{V}$ then $\mathcal{B}$ is a basis for $\mathcal{V}$. (b) Show that if $\mathcal{B}$ is a maximal linearly independent subset of $\mathcal{V}$ then $\mathcal{B}$ is a basis for $\mathcal{V}$.

2. Let $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, ..., \boldsymbol{b}_n\}$ be a basis for a vector space $\mathcal{V}$. Prove that each $\boldsymbol{v} \in \mathcal{V}$ can be expressed as a linear combination of the $\boldsymbol{b}_i$'s

$$\boldsymbol{v} = \alpha_1 \boldsymbol{b}_1 + \alpha_2 \boldsymbol{b}_2 + ... + \alpha_n \boldsymbol{b}_n,$$

in only one way - i.e., the *coordinates* α_i are unique.

Characterizations of a Basis Let $\mathcal{V}$ be a subspace of $\mathbb{R}^m$, and let $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, ..., \boldsymbol{b}_n\} \subseteq \mathcal{V}$. The following statements are equivalent.

- $\mathcal{B}$ is a basis for $\mathcal{V}$.
- $\mathcal{B}$ is a minimal spanning set for $\mathcal{V}$.
- $\mathcal{B}$ is a maximal linearly independent subset of $\mathcal{V}$.

Dimension The **dimension** of a vector space $\mathcal{V}$ is defined to be
$\dim\mathcal{V}$ = number of vectors in any basis for $\mathcal{V}$
= number of vectors in any minimal spanning set for $\mathcal{V}$
= number of vectors in any maximal independent subset of $\mathcal{V}$.

3. If $\mathcal{V}$ is an n-dimensional space, explain why every independent subset $\mathcal{S} = \{v_1, v_2, ..., v_n\} \subset \mathcal{V}$ containing n vectors must be a basis for $\mathcal{V}$.

4. For vector spaces $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \subseteq \mathcal{N}$. (a) Show that $\dim\mathcal{M} \leq \dim\mathcal{N}$. (b) If $\dim\mathcal{M} = \dim\mathcal{N}$, show that then $\mathcal{M} = \mathcal{N}$.

Subspace Dimension For vector spaces $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \subseteq \mathcal{N}$, the following statements are true.

- $\dim\mathcal{M} \leq \dim\mathcal{N}$.
- If $\dim\mathcal{M} = \dim\mathcal{N}$, then $\mathcal{M} = \mathcal{N}$.

Fundamental Subspaces – Dimension and Bases For an $m \times n$ matrix of real numbers such that $\operatorname{rank}(A) = r$,

- $\dim \operatorname{im}(A) = r$,
- $\dim \ker(A) = n - r$,
- $\dim \operatorname{im}(A^\top) = r$,
- $\dim \ker(A^\top) = m - r$.

Let P be a nonsingular matrix such that $PA = U$ is in row echelon form, and let $\mathcal{H}$ be the set of $\boldsymbol{h}_i$'s appearing in the general solution of $A\boldsymbol{x} = \boldsymbol{0}$.

- The basic columns of A form a basis for $\operatorname{im}(A)$.
- The nonzero rows of U form a basis for $\operatorname{im}(A^\top)$.
- The set $\mathcal{H}$ is a basis for $\ker(A)$.
- The last $m - r$ rows of P form a basis for $\ker(A^\top)$.

For matrices with complex entries, the above statements remain valid provided that $A^\top$ is replaced with $\overline{A}^\top$.

5. Find the dimensions of the four fundamental subspaces associated with $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}$.

[4] Historical note: The idea of defining a vector space by using a set of abstract axioms was contained in a general theory published in 1844 by Hermann Grassmann (1808 – 1887), a theologian and philosopher from Stettin, Poland, who was a self-taught mathematician. But Grassmann's work was originally ignored because he tried to construct a highly abstract self-contained theory, independent of the rest of mathematics, containing nonstandard terminology and notation, and he had a tendency to mix mathematics with obscure philosophy. Grassmann published a complete revision of his work in 1862 but with no more success. Only later was it realized that he had formulated the concepts we now refer to as linear dependence, bases, and dimension. The Italian mathematician Giuseppe Peano (1858–1932) was one of the few people who noticed Grassmann's work, and in 1888 Peano published a condensed interpretation of it. In a small chapter at the end, Peano gave an axiomatic definition of a vector space similar to the one from first lesson, but this drew little attention outside of a small group in Italy. The current definition is derived from the 1918 work of the German mathematician Hermann Weyl (1885–1955). Even though Weyl's definition is closer to Peano's than to Grassmann's, Weyl did not mention his Italian predecessor, but he did acknowledge Grassmann's "epoch making work." Weyl's success with the idea was due in part to the fact that he thought of vector spaces in terms of geometry, whereas Grassmann and Peano treated them as abstract algebraic structures. As we will see, it's the geometry that's important.

Rank Plus Nullity Theorem

- $\dim \operatorname{im}(A) + \dim \ker(A) = n$ for all $m \times n$ matrices.

6. Determine the dimensions of each of the following vector spaces: (a) The space of polynomials having degree n or less. (b) The space $\mathrm{Mat}_{m\times n}(\mathbb{R})$ of $m \times n$ matrices. (c) The space of $n \times n$ symmetric matrices.

7. (a) Determine the dimension as well as a basis for the space spanned by $\mathcal{S} = \left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}5\\6\\7\end{pmatrix} \right\}$.

(b) Determine the dimension of the space spanned by the set $\mathcal{S} = \left\{ \begin{pmatrix}1\\2\\-1\\3\end{pmatrix}, \begin{pmatrix}1\\0\\0\\2\end{pmatrix}, \begin{pmatrix}2\\8\\-4\\8\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}3\\3\\0\\6\end{pmatrix} \right\}$.

8. Determine whether or not the set $\mathcal{B} = \left\{ \begin{pmatrix}2\\3\\2\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix} \right\}$ is a basis for the space spanned by the set $\mathcal{A} = \left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}5\\8\\7\end{pmatrix}, \begin{pmatrix}3\\4\\1\end{pmatrix} \right\}$.

9. If $\mathcal{S}_r = \{v_1, v_2, ..., v_r\}$ is a linearly independent subset of an n-dimensional space $\mathcal{V}$, where $r < n$, explain why it must be possible to find extension vectors $\{v_{r+1}, ..., v_n\}$ from $\mathcal{V}$ such that

$$\mathcal{S}_n = \{v_1, v_2, ..., v_r, v_{r+1}, ..., v_n\}$$

is a basis for $\mathcal{V}$. As example, extend the independent set $\mathcal{S} = \left\{ \begin{pmatrix}1\\0\\-1\\2\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-2\end{pmatrix} \right\}$ to a basis for $\mathbb{R}^4$.

10. Consider the following matrix and column vector: $A = \begin{pmatrix}1&2&2&0&5\\2&4&3&1&8\\3&6&1&5&5\end{pmatrix}$ and $v = (-8,1,3,3,0)^\top$. Verify that $v \in \ker(A)$, and then extend $\{v\}$ to a basis for $\ker(A)$.

11. Construct a 4×4 homogeneous system of equations that has no zero coefficients and three linearly independent solutions.

Dimension of a Sum If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of a vector space $\mathcal{V}$, then

$$\dim(\mathcal{X} + \mathcal{Y}) = \dim\mathcal{X} + \dim\mathcal{Y} - \dim(\mathcal{X} \cap \mathcal{Y}).$$

12. (a) Show that $\operatorname{rank}(A + B) \le \operatorname{rank}(A) + \operatorname{rank}(B)$. (b) Explain why $|\operatorname{rank}(A) - \operatorname{rank}(B)| \le \operatorname{rank}(A - B)$.

13. For $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ and a subspace $\mathcal{S}$ of $\mathbb{R}^n$, the image

$$A(\mathcal{S}) = \{A\boldsymbol{x} \mid \boldsymbol{x} \in \mathcal{S}\}$$

of $\mathcal{S}$ under A is a subspace of $\mathbb{R}^m$. Prove that if $\mathcal{S} \cap \ker(A) = 0$, then $\dim A(\mathcal{S}) = \dim(\mathcal{S})$.

14. Explain why every nonzero subspace $\mathcal{V} \subseteq \mathbb{R}^n$ must possess a basis.

15. Let $A, E \in \mathrm{Mat}_{m\times n}(\mathbb{R})$. If $\operatorname{rank}(A) = r$ and $\operatorname{rank}(E) = k \le r$, explain why

$$r - k \le \operatorname{rank}(A + E) \le r + k.$$

In words, this says that a perturbation of rank k can change the rank by at most k.

16. Let $\mathcal{V}$ denote vector space of all matrices of form 2×2 over the field of real numbers. Let $\mathcal{W}_1$ be the set of all matrices of form $\begin{pmatrix}x&-x\\y&z\end{pmatrix}$ and let $\mathcal{W}_2$ be the set of all matrices of the form $\begin{pmatrix}a&b\\-a&c\end{pmatrix}$. Find a basis and the dimensions of the four subspaces $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_1 + \mathcal{W}_2$ and $\mathcal{W}_1 \cap \mathcal{W}_2$.

17. Show that $\mathcal{V} = \{A \in \mathrm{Mat}_{2\times2}(\mathbb{R}) \mid \operatorname{trace}(A) = 0\}$ is subspace of a vector space $\mathrm{Mat}_{2\times2}(\mathbb{R})$ (where $\operatorname{trace}(A) =$ sum of diagonal entries of A). Find a basis and the dimension. Basis that you get extend to full basis of $\mathrm{Mat}_{2\times2}(\mathbb{R})$.

18. In space of all real sequences $\mathbb{R}^{\mathbb{N}}$ ($\mathbb{R}^{\mathbb{N}} = \{(a_1, a_2, a_3, ..., a_n, a_{n+1}, ...) \mid a_n \in \mathbb{R}, n \in \mathbb{N}\}$) let $\mathcal{L}$ be a given set

$$\mathcal{L} = \{(a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \mid a_{n+2} - 2a_n = 0, n \in \mathbb{N}\}.$$

Show that $\mathcal{L}$ is subspace of $\mathbb{R}^{\mathbb{N}}$ and find its basis and the dimension.

19. **(IMC 2012.)** Let $n \ge 3$ be a fixed positive integer, and let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ denote a matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal. Determine the smallest possible number r such that

$$\dim(\operatorname{im}(A)) = r.$$

For r that you get, give an example of matrix.

InC: 3, 6, 7(a), 9, 11, 12(a), 13. HW: 16, 17, 18, 19 + three problems from the web page `http://osebje.famnit.upr.si/~penjic/linearnaAlgebra/`.